On the spectrum of a waveguide with periodic cracks

Konstantin Pankrashkin

Laboratoire de mathématiques d'Orsay (CNRS UMR 8628)
Université Paris-Sud, Bâtiment 425
91400 Orsay, France

Dedicated to the memory of Pierre Duclos

Abstract

The spectral problem on a periodic domain with cracks is studied. An asymptotic form of dispersion relations is calculated under assumption that the opening of the cracks is small.

Let $\Omega \subset \mathbf{R}^2$ be a connected smooth domain satisfying the following conditions:

- Ω is invariant under the shift $(x, y) \mapsto (x + 1, y)$,
- the domains $\Omega_n := \Omega \cap \Big([n, n+1] \times \mathbf{R}\Big)$, $n \in \mathbf{Z}$, are bounded,
- the points $A_n := (n, 0)$ are interior points of Ω.

Assume that the vertical lines $L_n = (n, \mathbf{R})$, $n \in \mathbf{Z}$, are non-tangent to the boundary of Ω and denote $\Gamma_n^\varepsilon := L_n \cap \Omega \setminus B_\varepsilon(A_n)$, where $B_\varepsilon(A_n)$ is the ball of radius ε centered at A_n.

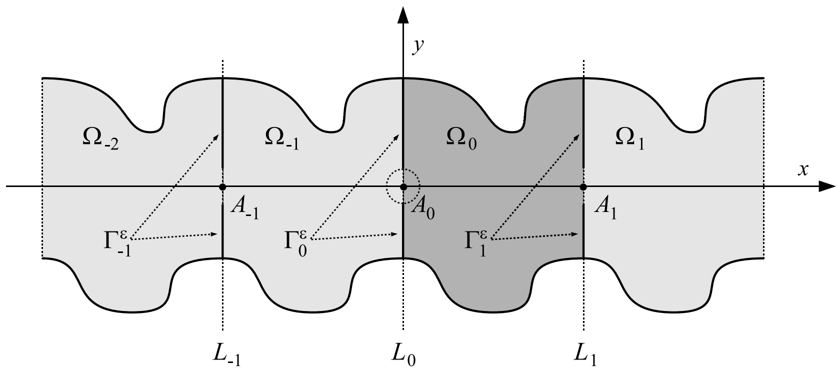

Figure 1. Domain Ω

Consider in $L^2(\Omega)$ the operator H_ε which is the Laplacian with the Neumann boundary conditions on $\partial\Omega$ and on all the curves Γ_n^ε. We are studying some spectral properties of H_ε as ε tends to 0.

The operator H_ε can be viewed as the Hamiltonian of elementary cells Ω_n coupled by small windows. For positive ε one has a periodic system with a band spectrum, while at $\varepsilon = 0$ the system is decoupled. There is a number of papers concerning the spectral asymptotics for waveguides with windows or cracks in various settings, see e.g. [3–5,7–9]. In particular, the paper [9] studied a situation which is close to ours, more precisely, the case of the Dirichlet boundary conditions on a similar periodic structure with particular symmetries (straight strip with symmetric cuts), and the asymptotics of the lowest dispersion relation was calculated. All the papers cited use the method of matching of asymptotic expansions [6], which is quite sensible to the geometric properties. We give here another proof based on the analysis of operator pencils involving boundary integrals and inspired by the recent monograph [1]. This allows one to obtain the asymptotics of the spectrum in rather general situations (the assumption made above can be relaxed is various directions, we have just chosen a basic situation in order to keep simple notations). Integral operator pencils of a similar type were used by Pierre Duclos with co-authors as a part of the skeleton method for the study of multi-particle problems [2].

By periodicity, the spectrum of H_ε can be studied with the help of the Bloch theory. For $\theta \in [0, 2\pi]$ consider the values $E_n(\theta; \varepsilon)$ for which there exists a non-zero function $\Psi_n(x, y; \theta, \varepsilon)$ satisfying the equation $-\Delta\Psi_n(x, y; \theta, \varepsilon) = E_n(\theta, \varepsilon)\Psi_n(x, y; \theta, \varepsilon)$, the above boundary conditions, and the Bloch quasiperiodicity condition $\Psi_n(x + 1, y; \theta, \varepsilon) = e^{i\theta}\Psi_n(x, y; \theta, \varepsilon)$; here $\theta \in [0, 2\pi]$ is a real number called quasimomentum. By a suitable reordering one can assume that the functions $\theta \to E_n(\theta, \varepsilon)$ are continuous and 2π-periodic; these function are called dispersion relations of H_ε. The image of a dispersion relation is called a spectral band of H_ε, and the spectrum of H_ε is the union of the spectral bands.

We are interested in the asymptotic form of the dispersion relations for small ε. By the standard variational arguments, see e.g. Proposition 1.3 in [9] or proposition 9.8 in [3], each function $E_n(\cdot, \varepsilon)$ converges to a constant function equal to an eigenvalue of the decoupled system, i.e. to an eigenvalue of the Neumann Laplacian on Ω_0. Our main result is the following theorem.

Theorem 1 *Let E be a simple eigenvalue of the Neumann Laplacian in Ω_0 and u be the corresponding normalized eigenfunction. As $\varepsilon \to 0$, the dispersion relation of H_ε near E is of the form*

$$E(\theta, \varepsilon) = E + \frac{2\pi}{|\log \varepsilon|}\big|\, u(A_0) - e^{i\theta}u(A_1)\big|^2 + O\Big(\frac{1}{\log^2 \varepsilon}\Big)$$

uniformly in $\theta \in [0, 2\pi]$.

We note that E can be *any* simple eigenvalue, and not just the bottom one.

The rest of the paper is devoted to the proof of theorem 1.

Let $G_n(r, r'; z)$ be the Green function of the Neumann Laplacian N_n in Ω_n, i.e. the integral kernel of the resolvent $(N_n - z)^{-1}$. Recall that, if z is not an eigenvalue of N_n, then the boundary value problem

$$(-\Delta - z)f = 0 \text{ in } \Omega_n, \quad \frac{\partial f}{\partial \nu}\Big|_{\partial\Omega_n} = g,$$

where ν is the exterior normal vector, has the unique solution

$$f(r) = \int_{\partial\Omega_n} G_n(r, r'; z) g(r')\, dl(r').$$

Consider a Bloch solution Ψ of H_ε corresponding a quasimomentum θ and to an eigenvalue $z = E(\theta)$. Introduce the functions

$$f_n(y) := \frac{\partial \Psi(x, y)}{\partial x}\Big|_{x=n}, \quad y \in (-\varepsilon, \varepsilon).$$

Denote by Ψ_n the restriction of Ψ to Ω_n. One has, obviously,

$$\Psi_n(x, y) = -\int_{-\varepsilon}^{\varepsilon} G_n(x, y; n, y'; z) f_n(y') dy' + \int_{-\varepsilon}^{\varepsilon} G_n(x, y; n+1, y'; z) f_{n+1}(y') dy'.$$

The Bloch condition for Ψ implies $f_{n+1}(y') = e^{i\theta} f_n(y')$. The continuity of Ψ at $x = n$ takes the form $\Psi_n(n, y) = \Psi_{n-1}(n, y)$, or, using the above integral representation,

$$\begin{aligned} &e^{i\theta} \int_{-\varepsilon}^{\varepsilon} G_n(n, y; n+1, y'; z) f_n(y') dy' - \int_{-\varepsilon}^{\varepsilon} G_n(n, y; n, y'; z) f_n(y') dy' \\ &= \int_{-\varepsilon}^{\varepsilon} G_{n-1}(n, y; n, y'; z) f_n(y') dy' - e^{-i\theta} \int_{-\varepsilon}^{\varepsilon} G_{n-1}(n, y; n-1, y'; z) f_n(y') dy'. \end{aligned}$$

Using the obvious identity $G_n(x, y; x', y'; z) = G_0(x - n, y; x' - n, y'; z)$ one arrives at a single integral equation

$$K_{\theta,\varepsilon}(z) f_{\theta,\varepsilon} := \int_{-\varepsilon}^{\varepsilon} K_{\theta,\varepsilon}(y, y'; z) f_{\theta,\varepsilon}(y')\, dy' = 0, \quad f_{\theta,\varepsilon} := f_0, \tag{1}$$

with the integral kernel

$$\begin{aligned} K_{\theta,\varepsilon}(y, y'; z) &= G_0(0, y; 0, y'; z) + G_0(1, y; 1, y'; z) \\ &\quad - e^{i\theta} G_0(0, y; 1, y'; z) - e^{-i\theta} G_0(1, y; 0, y'; z). \end{aligned}$$

In order to determine the dispersion relations of H_ε one needs to find the values of z for which the equation (1) has non-zero solutions, i.e. the nonlinear eigenvalues of $K_{\theta,\varepsilon}$.

As noted above, integral equations of this type were studied in details in the recent monograph [1], and here we add some details needed for the treatment of the periodic problem. First of all, by Lemma 5.6 in [1], for each fixed θ, the equation (1) has a unique solution in a neighborhood of E if ε is small enough.

Like in [1] introduce the Hilbert space

$$X_\varepsilon := \big\{\varphi :\ \|\varphi\|^2_{X_\varepsilon} := \int_{-\varepsilon}^{\varepsilon} \sqrt{\varepsilon^2 - y^2} |\varphi(y)|^2\, dy < +\infty\big\},$$

and the linear space

$$Y_\varepsilon := \{\psi \in C[-\varepsilon, \varepsilon] : \psi' \in X_\varepsilon\}.$$

By the trace theorems, the functions $f_{\theta,\varepsilon}$ belong to both X_ε and Y_ε.

We emphasize that the symbol $\langle\cdot,\cdot\rangle$ will *always* denote the usual scalar product in $L^2[-\varepsilon,\varepsilon]$, i.e.

$$\langle f,g\rangle := \int_{-\varepsilon}^{\varepsilon} \overline{f(y)}\, g(y)\, dy.$$

Using Lemma 5.1 in [1] and the spectral representation of the Green function G_0 one can decompose $K_{\theta,\varepsilon}$ as follows:

$$K_{\theta,\varepsilon}(z) := -\frac{1}{2\pi} L_\varepsilon + \frac{M_{\theta,\varepsilon}}{E-z} + R_{\theta,\varepsilon}(z), \tag{2}$$

where L_ε is the integral operator

$$L_\varepsilon f(x) = \int_{-\varepsilon}^{\varepsilon} \log|x-y|\, f(y)\, dy,$$

$M_{\theta,\varepsilon}$ is the rank one operator

$$M_{\theta,\varepsilon} f(x) = u_{\theta,\varepsilon}(x) \int_{-\varepsilon} \overline{u_{\theta,\varepsilon}(y)}\, f(y)\, dy, \quad u_{\theta,\varepsilon}(y) := u(0,y) - e^{i\theta} u(1,y),$$

and $R_{\theta,\varepsilon}(z)$ is an integral operator

$$R_{\theta,\varepsilon}(z) f(x) = \int_{-\varepsilon}^{\varepsilon} R_{\theta,\varepsilon}(x,y;z)\, f(y)\, dy,$$

where the kernel $R_{\theta,\varepsilon}(x,y;z)$ is holomorph in (z,θ) and of Hölder class $C^{1+\alpha}$ with respect to (x,y); here $\alpha\in(0,1)$ is arbitrary. Note that the functions $u_{\theta,\varepsilon}$ are in both X_ε and Y_ε as well.

By Lemma 5.2 in [1], the operator $L_\varepsilon : X_\varepsilon \to Y_\varepsilon$ is a bijection for ε small enough, and, by Lemma 5.4 in [1], one has, uniformly in $\theta\in[0,2\pi]$ and z in a neighborhhod of E, the norm estimate

$$\|L_\varepsilon^{-1} R_{\theta,\varepsilon}(z)\|_{L(X_\varepsilon,X_\varepsilon)} = O\Big(\frac{1}{\log\varepsilon}\Big); \tag{3}$$

here and below $L(X_\varepsilon,X_\varepsilon)$ is the Banach space of bounded linear operators on X_ε. We will need the following estimates:

Proposition 2 *As $\varepsilon\to 0$ one has, uniformly in θ,*

$$\langle L_\varepsilon^{-1} u_{\theta,\varepsilon}, u_{\theta,\varepsilon}\rangle = \frac{\big|u_{\theta,\varepsilon}(0)\big|^2}{\log\varepsilon} + O\Big(\frac{1}{|\log\varepsilon|^2}\Big). \tag{4}$$

Futhermore, there exists $A>0$ with the following property: If B is a bounded operator in X_ε then, as $\varepsilon\to 0$, one has

$$\big|\langle B L_\varepsilon^{-1} u_{\theta,\varepsilon}, u_{\theta,\varepsilon}\rangle\big| \le \frac{A\|B\|_{L(X_\varepsilon,X_\varepsilon)}}{|\log\varepsilon|}. \tag{5}$$

Proof As in Lemma 5.4 of [1] we use an explicit form for the inverse of L_ε and some properties of the finite Hilbert transform. We have

$$\begin{aligned} L_\varepsilon^{-1} u_{\theta,\varepsilon}(x) = &-\frac{1}{\pi^2\sqrt{\varepsilon^2-x^2}} \int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-y^2}\, u'_{\theta,\varepsilon}(y)}{x-y} dy \\ &+ \frac{a(u_{\theta,\varepsilon})}{\pi\log\frac{\varepsilon}{2}} \cdot \frac{1}{\sqrt{\varepsilon^2-x^2}} =: I_1 + I_2, \end{aligned}$$

where

$$a(u_{\theta,\varepsilon}) = u_{\theta,\varepsilon}(x) - L_\varepsilon v_{\theta,\varepsilon}(x)$$

(the choice of x is arbitrary) with

$$v_{\theta,\varepsilon}(x) = -\frac{1}{\pi^2\sqrt{\varepsilon^2 - x^2}} \int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2 - y^2}\, u'_{\theta,\varepsilon}(y)}{x-y} dy;$$

here and below all the integrals are understood in the sence of the Cauchy principal value.

We will use the following well known estimate: for any $\alpha \in (0,1)$ there exists $C > 0$ such that for any $\varphi \in C^\alpha[-1,1]$ and $x \in [-1,1]$ there holds

$$\Big| \int_{-1}^{1} \frac{\varphi(y)}{x-y} dy \Big| \le C \|\varphi\|_{C^\alpha[-1,1]}.$$

One has then

$$\begin{aligned}
\|I_1\|_{X_\varepsilon} &= \frac{1}{\pi^2} \sqrt{\int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{\varepsilon^2 - x^2}} \Big(\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2 - y^2} u'_{\theta,\varepsilon}(y)}{x-y} dy \Big)^2 dx} \\
&\le \frac{1}{\pi^2} \sqrt{\int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{\varepsilon^2 - x^2}} dx}\ \Big\| \int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2 - y^2} u'_{\theta,\varepsilon}(y)}{x-y}\, dy \Big\|_{L^\infty[-\varepsilon,\varepsilon]} \\
&= \frac{\varepsilon}{\pi^{3/2}} \Big\| \int_{-1}^{1} \frac{\sqrt{1-y^2}\, u'_{\theta,\varepsilon}(\varepsilon y)}{x-y}\, dy \Big\|_{L^\infty[-1,1]} \\
&\le \frac{C\varepsilon}{\pi^{3/2}} \Big\| \sqrt{1-y^2}\, u'_{\theta,\varepsilon}(\varepsilon y) \Big\|_{C^\alpha[-1,1]},
\end{aligned} \tag{6}$$

and one obtains $\|I_1\|_{X_\varepsilon} = O(\varepsilon)$ uniformly in θ. This implies $\langle I_1, u_{\theta,\varepsilon}\rangle = O(\varepsilon)$ as well. Hence it is sufficient to study the asymptotics of $\langle I_2, u_{\theta,\varepsilon}\rangle$. We have

$$\begin{aligned}
\langle I_2, u_{\theta,\varepsilon}\rangle &= \frac{1}{\pi \log|\varepsilon/2|} \Big(\int_{-\varepsilon}^{\varepsilon} \frac{\big|u_{\theta,\varepsilon}(x)\big|^2}{\sqrt{\varepsilon^2 - x^2}} dx - \int_{-\varepsilon}^{\varepsilon} \overline{L_\varepsilon v_{\theta,\varepsilon}(y)}\, \frac{u_{\theta,\varepsilon}(y)}{\sqrt{\varepsilon^2 - y^2}}\, dy \Big) \\
&=: \frac{1}{\pi \log|\varepsilon/2|} (K_1 - K_2).
\end{aligned}$$

There holds

$$\begin{aligned}
K_1 &= \int_{-\varepsilon}^{\varepsilon} \frac{\big|u_{\theta,\varepsilon}(x)\big|^2}{\sqrt{\varepsilon^2 - x^2}} dx = \int_{-1}^{1} \frac{\big|u_{\theta,\varepsilon}(\varepsilon x)\big|^2}{\sqrt{1-x^2}} dx \\
&= \big|u_{\theta,\varepsilon}(0)\big|^2 \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} + \int_{-1}^{1} \frac{\big|u_{\theta,\varepsilon}(\varepsilon x)\big|^2 - \big|u_{\theta,\varepsilon}(0)\big|^2}{\sqrt{1-x^2}} dx \\
&= \pi \big|u_{\theta,\varepsilon}(0)\big|^2 + O(\varepsilon),
\end{aligned}$$

where the remainder term is uniform in θ.

To estimate K_2 let us note first that, by the same arguments as in (6),

$$\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2 - y^2}}{x-y} u'_{\theta,\varepsilon}(y)\, dy =: K_3(x) = O(\varepsilon)$$

uniformly in x and θ. Therefore,

$$\begin{aligned}\sup_{x\in[-\varepsilon,\varepsilon]}\big|L_\varepsilon v_{\theta,\varepsilon}(x)\big| &= \frac{1}{\pi^2}\sup_{x\in[-\varepsilon,\varepsilon]}\Big|\int_{-\varepsilon}^{\varepsilon}\log|x-y|\cdot K_3(y)\cdot\frac{1}{\sqrt{\varepsilon^2-y^2}}\,dy\Big|\\ &\le \frac{1}{\pi^2}\sup_{y\in[-\varepsilon,\varepsilon]}\big|K_3(y)\big|\sup_{x\in[-\varepsilon,\varepsilon]}\int_{-\varepsilon}^{\varepsilon}\frac{\big|\log|x-y|\big|}{\sqrt{\varepsilon^2-y^2}}dy\\ &= \frac{1}{\pi}\sup_{y\in[-\varepsilon,\varepsilon]}|K_3(y)|\,\log\frac{\varepsilon}{2} = O(\varepsilon\log\varepsilon)\end{aligned}$$

uniformly in θ. Finally,

$$|K_2| \le \sup_{x\in[-\varepsilon,\varepsilon]}\big|L_\varepsilon v_{\theta,\varepsilon}(x)\big|\int_{-\varepsilon}^{\varepsilon}\frac{\big|u_{\theta,\varepsilon}(y)\big|}{\sqrt{\varepsilon^2-y^2}}\,dy = O(\varepsilon\log\varepsilon).$$

We obtain

$$\begin{aligned}\langle L_\varepsilon^{-1}u_{\theta,\varepsilon},u_{\theta,\varepsilon}\rangle &= \langle I_1,u_{\theta,\varepsilon}\rangle + \langle I_2,u_{\theta,\varepsilon}\rangle = \langle I_1,u_{\theta,\varepsilon}\rangle + \frac{K_1}{\pi\log\frac{\varepsilon}{2}} - \frac{K_2}{\pi\log\frac{\varepsilon}{2}}\\ = O(\varepsilon) + \frac{\pi\big|u_{\theta,\varepsilon}(0)\big|^2+O(\varepsilon)}{\pi\log\frac{\varepsilon}{2}} &+ \frac{O(\varepsilon\log\varepsilon)}{\pi\log\frac{\varepsilon}{2}} = \frac{\big|u_{\theta,\varepsilon}(0)\big|^2}{\log\varepsilon} + O\Big(\frac{1}{|\log\varepsilon|^2}\Big),\end{aligned}$$

which proves (4).

To show (5) we use first the Cauchy-Schwartz inequality for X_ε,

$$\begin{aligned}\big|\langle BL_\varepsilon^{-1}u_{\theta,\varepsilon},u_{\theta,\varepsilon}\rangle\big| &= \Big|\int_{-\varepsilon}^{\varepsilon}\sqrt{\varepsilon^2-x^2}\cdot\overline{BL_\varepsilon^{-1}u_{\theta,\varepsilon}(x)}\,\frac{u_{\theta,\varepsilon}(x)}{\sqrt{\varepsilon^2-x^2}}dx\Big|\\ &\le \|B\|_{L(X_\varepsilon,X_\varepsilon)}\cdot\|L_\varepsilon^{-1}u_{\theta,\varepsilon}\|_{X_\varepsilon}\cdot\sqrt{\int_{-\varepsilon}^{\varepsilon}\frac{\big|u_{\theta,\varepsilon}(x)\big|^2}{\sqrt{\varepsilon^2-x^2}}dx}\\ &\le a\|B\|_{L(X_\varepsilon,X_\varepsilon)}\|L_\varepsilon^{-1}u_{\theta,\varepsilon}\|_{X_\varepsilon},\quad a>0,\end{aligned}$$

and the previous estimates show that $\|L_\varepsilon^{-1}u_{\theta,\varepsilon}\|_{X_\varepsilon} = O(1/\log\varepsilon)$. Proposition 2 is proved.

Let us conclude the proof of theorem 1. Assume that $K_{\theta,\varepsilon}(z)f_{\theta,\varepsilon}(z) = 0$ with $f_{\theta,\varepsilon}\ne 0$ and $z = E(\theta,\varepsilon)$ sufficiently close to E. Using the invertibility of L_ε and the decomposition (2) one obtains

$$\Big(1-2\pi L_\varepsilon^{-1}R_{\theta,\varepsilon}\Big)f_{\theta,\varepsilon} = \frac{2\pi}{E-z}\,\langle u_{\theta,\varepsilon},f_{\theta,\varepsilon}\rangle L_\varepsilon^{-1}u_{\theta,\varepsilon}.$$

The operator $1-2\pi L_\varepsilon^{-1}R_{\theta,\varepsilon}$ is invertible for ε small enough, therefore,

$$f_{\theta,\varepsilon} = \frac{2\pi}{E-z}\big(1-2\pi L_\varepsilon^{-1}R_{\theta,\varepsilon}\big)^{-1}\langle u_{\theta,\varepsilon},f_{\theta,\varepsilon}\rangle L_\varepsilon^{-1}u_{\theta,\varepsilon},$$

which implies $\langle u_{\theta,\varepsilon},f_{\theta,\varepsilon}\rangle\ne 0$ (otherwise one would get $f_{\theta,\varepsilon}\equiv 0$). Taking the scalar product $\langle\cdot,\cdot\rangle$ of the both parts with $u_{\theta,\varepsilon}$ one arrives at

$$z = E(\theta,\varepsilon) = E - 2\pi\Big\langle\big(1-2\pi L_\varepsilon^{-1}R_{\theta,\varepsilon}\big)^{-1}L_\varepsilon^{-1}u_{\theta,\varepsilon},u_{\theta,\varepsilon}\Big\rangle.$$

Using (3), let us fix $B > 0$ such that

$$\|L_\varepsilon^{-1} R_{\theta,\varepsilon}(z)\|_{L(X_\varepsilon, X_\varepsilon)} \leq \frac{B}{|\log \varepsilon|} \tag{7}$$

for small ε. As the map $f \mapsto \langle f, u_{\theta,\varepsilon}\rangle$ is continuous in the X_ε-norm, we have, for sufficiently small ε such that $\frac{2\pi B}{|\log \varepsilon|} < 1$,

$$\begin{aligned} &\Big\langle \big(1 - 2\pi L_\varepsilon^{-1} R_{\theta,\varepsilon}\big)^{-1} L_\varepsilon^{-1} u_{\theta,\varepsilon}, u_{\theta,\varepsilon}\Big\rangle \\ &= \langle L_\varepsilon^{-1} u_{\theta,\varepsilon}, u_{\theta,\varepsilon}\rangle + \sum_{k\geq 1} \Big\langle \big(2\pi L_\varepsilon^{-1} R_{\theta,\varepsilon}\big)^k L_\varepsilon^{-1} u_{\theta,\varepsilon}, u_{\theta,\varepsilon}\Big\rangle =: D_1 + D_2. \end{aligned}$$

In view of the estimate (4) for D_1, it is sufficient show that $D_2 = O\big(1/|\log \varepsilon|^2\big)$. By (3) and (5),

$$\Big|\Big\langle \big(2\pi L_\varepsilon^{-1} R_{\theta,\varepsilon}\big)^k L_\varepsilon^{-1} u_{\theta,\varepsilon}, u_{\theta,\varepsilon}\Big\rangle\Big| \leq \frac{A}{|\log \varepsilon|}\Big(\frac{2\pi B}{|\log \varepsilon|}\Big)^k,$$

hence

$$|D_2| \leq \frac{A}{|\log \varepsilon|} \sum_{k\geq 1} \Big(\frac{B}{|\log \varepsilon|}\Big)^k = \frac{A}{|\log \varepsilon|} \cdot \frac{\frac{2\pi B}{|\log \varepsilon|}}{1 - \frac{2\pi B}{|\log \varepsilon|}} = O(1/|\log \varepsilon|^2),$$

which finishes the proof of theorem 1.

References

[1] H. Ammari, H. Kang, H. Lee: *Layer Potential Techniques in Spectral Analysis* (Volume 153 of the "Mathematical Surveys and Monographs" series, AMS, 2009).

[2] H. D. Cornean, P. Duclos, B. Ricaud: *On the skeleton method and an application to a quantum scissor.* In the book *Analysis on graphs and its applications* (Volume 77 of the "Proceedings of Symposia in Pure Mathematics" series, AMS, Providence, 2008) 657–672.

[3] M. Dauge, B. Helffer: *Eigenvalues variation. II. Multidimensional problems.* J. Differential Equations **104** (1993) 263–297.

[4] P. Exner, S. A. Vugalter: *Asymptotic estimates for bound states in quantum waveguides coupled laterally through a narrow window.* Ann. Inst. Henri Poincaré **65** (1996) 109–123.

[5] R. Gadyl'shin: *On the Dirichlet problem for the Helmholtz equation in the plane with boundary conditions on an almost closed curve.* Sb. Math. **191** (2000) 821–848.

[6] A. M. Il'in: *Matching of asymptotic expansions of solutions of boundary value problems* (Volume 102 of the "Translations of mathematical monographs" series, AMS, 1992).

[7] I. Yu. Popov: *Asymptotics of bound states and bands for laterally coupled waveguides and layers.* J. Math. Phys. **43** (2002) 215–234.

[8] K. Yoshitomi: *Eigenvalue Problems on Domains with Cracks I.* Tokyo J. of Math. **28** (2005) 341–380.

[9] K. Yoshitomi: *Band spectrum of the Laplacian on a slab with the Dirichlet boundary condition on a grid.* Kyushu J. Math. **57** (2003) 87–116.